Group-wise Median vs Element-wise Median

Given a discrete set $X$ partitioned into $N$ subsets $X_i$, define $n_i$ to be the number of elements in partition $X_i$, aka the size$^1$ of $X_i$. Without loss of generality, we can label the partitions $X_i$ in such a way that

$$n_0 \leq n_1 \leq \cdots \leq n_i \leq n_{i+1} \leq \cdots \leq n_N,$$

which then allows us to order the partitions.

The median partition, that is, the partition for which half of the partitions are bigger and half are smaller, is then $X_{N/2}$ (for ease of notation we will assume for everything that follows that $N$ is even; if $N$ is odd the argument follows along similar lines).

The element-wise median partition is the partition for which “half” of the elements of $X$ are in bigger partitions and “half” of the elements are in smaller partitions. Specifically, it is the partition $X_m$ such that

$$f(m) \equiv \frac{\sum_{i=0}^{m} n_i}{\sum_{i=0}^{N} n_i} \geq \frac{1}{2},$$

but $f(m + 1) < \frac{1}{2}$.

**Theorem**  We now prove that the element-wise median is always at least as large as the median, that is, that $m \geq \frac{N}{2}$.

To start, it is clear from the size-ordering of the partitions that

$$\sum_{i=0}^{N} n_i \geq \sum_{i=0}^{N/2} n_i,$$

from which it naturally follows that

$$2 \cdot \sum_{i=0}^{N} n_i \geq \sum_{i=0}^{N/2} n_i + \sum_{i=N/2}^{N} n_i = \sum_{i=0}^{N} n_i.$$

Dividing both sides by $2 \cdot \sum_{i=0}^{N} n_i$ yields the desired expression:

$$\frac{\sum_{i=0}^{N/2} n_i}{\sum_{i=0}^{N} n_i} \geq \frac{1}{2},$$

or equivalently, $f(\frac{N}{2}) \geq \frac{1}{2}$. It clearly follows that $m$ cannot be less than $\frac{N}{2}$, else $f(m+1) \geq \frac{1}{2}$ which violates the definition of $m$. Thus $m \geq \frac{N}{2}$.

$^1$note: $n_i$ is related to the “frequency” of a data point, that is, if we create a new set $F$ from $X$ by replacing each $x \in X$ with the size $n_i$ of the partition it belongs to: $F \equiv \{n_i(x) | x \in X\}$, then $n_i$ is its own frequency, that is, there are $n_i$ elements of value $n_i$ in $F$. \hfill \square