Group-wise Median vs Element-wise Median

Given a discrete set X partitioned into N subsets X_i, define n_i to be the number of elements in partition X_i, aka *the size*[1] of X_i. Without loss of generality, we can label the partitions X_i in such a way that

$$n_0 \leq n_1 \leq \cdots \leq n_i \leq n_{i+1} \leq \cdots \leq n_N,$$

which then allows us to order the partitions.

The **median** partition, that is, the partition for which half of the partitions are bigger and half are smaller, is then $X_{N/2}$ (for ease of notation we will assume for everything that follows that N is even; if N is odd the argument follows along similar lines).

The **element-wise median** partition is the partition for which "half" of the elements of X are in bigger partitions and "half" of the elements are in smaller partitions. Specifically, it is the partition X_m such that

$$f(m) \equiv \frac{\sum_{i=m}^{N} n_i}{\sum_{i=0}^{N} n_i} \geq \frac{1}{2},$$

but $f(m+1) < \frac{1}{2}$.

Theorem We now prove that the element-wise median is always at least as large as the median, that is, that $m \geq \frac{N}{2}$.

To start, it is clear from the size-ordering of the partitions that

$$\sum_{i=\frac{N}{2}}^{N} n_i \geq \sum_{i=0}^{\frac{N}{2}-1} n_i,$$

from which it naturally follows that

$$2 \cdot \sum_{i=\frac{N}{2}}^{N} n_i \geq \sum_{i=0}^{\frac{N}{2}-1} n_i + \sum_{i=\frac{N}{2}}^{N} n_i = \sum_{i=0}^{N} n_i.$$

Dividing both sides by $2 \cdot \sum_{i=0}^{N} n_i$ yields the desired expression:

$$\frac{\sum_{i=\frac{N}{2}}^{N} n_i}{\sum_{i=0}^{N} n_i} \geq \frac{1}{2},$$

or equivalently, $f\left(\frac{N}{2}\right) \geq \frac{1}{2}$. It clearly follows that m cannot be less than $\frac{N}{2}$, else $f(m+1) \geq \frac{1}{2}$ which violates the definition of m. Thus $m \geq \frac{N}{2}$. $\square$

[1] note: n_i is related to the "frequency" of a data point, that is, if we create a new set F from X by replacing each $x \in X$ with the size n_i of the partition it belongs to: $F \equiv \{n_i(x) | x \in X\}$; then n_i is its own frequency, that is, there are n_i elements of value n_i in F.